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Solution to the Schrödinger Equation for the Time-Dependent Potential

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Abstract In this work, the Schrödinger equation with the time-dependent potential $V(z,\hat{p},t) = g_1(t)z + g_2(t)\hat{p} + g_3(t)$ has been solved by the method of time-space transformation in $1+1$ dimensions. The corresponding analytical wave function to Schrödinger equation is obtained. In addition, the discussion of solutions to particular cases has been made.

Keywords Schrödinger equation · Time-dependent potential · Analytical wave function

1 Introduction

It is well known that the time-dependent quantum-mechanical problems [1, 2] have attracted the considerable interest of physicists . Beside the mathematical interest, the knowledge of the solution may help us to further explore various fascinating quantum phenomena and clarify some subtle concepts. Its study has been applied widely in the analyses of quantum fields in curved space-time [3], quantum optic [4, 5], the Pauli trap [6–8] and spintronics [9]. Over past decades, an extensive effort has been made to find the analytical wave function of harmonic oscillator with time-dependent mass or frequencies by different methods such as: (i) path integral [10], (ii) dynamical invariant [10–13], and (iii) second quantization [10]. Beside the time-dependent harmonic oscillator, recently there has been particular attention in the solving one-dimensional Schrödinger equation [14–16] and Dirac equation [17, 18] with time-dependent linear potential, using the technique of Lewis and Riesenfeld and the time-space transformation method, respectively. In the present work, our main purpose is to obtain, through the method of time-space transformation, the exact solution to Schrödinger

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equation with the time-dependent potential: $V(z,\hat{p},t)=g_1(t)z+g_2(t)\hat{p}+g_3(t)$, ($g_j(t)$ being real function of time, $j=1,2,3$), in $1+1$ dimensions. In addition, the solutions to particular cases have been discussed.

2 The solution of Schrödinger Equation for Time-Dependent Potential

In $1+1$ dimensions, Schrödinger equation with time-dependent mass $m(t)$ and time-dependent potential $V(z,\hat{p},t)=g_1(t)z+g_2(t)\hat{p}+g_3(t)$ reads

$$\left\{\frac{\partial}{\partial t}-\frac{1}{i}\hat{H}\right\}\psi(z,t)=0,\quad \hat{H}=\frac{\hat{p}^2}{2m(t)}+g_1(t)z+g_2(t)\hat{p}+g_3(t), \tag{1}$$

where we have used $\hbar=1$, for the sake of simplicity. Performing a unitary transformation:

$$\psi(z,t)=\varphi(z,t)e^{i\beta(t)z},$$

where $\beta(t)$ determined later is a real function of time, (1) can be written in the following form:

$$i\frac{\partial}{\partial t}\varphi=-\frac{1}{2m}\frac{\partial^2}{\partial z^2}\varphi-i\left\{\frac{\beta}{m}+g_2\right\}\frac{\partial\varphi}{\partial z}+\left\{\frac{\beta^2}{2m}+\beta g_2+g_3\right\}\varphi+\{g_1+\dot{\beta}\}z\varphi, \tag{2}$$

where the dot on the arbitrary variables denotes the derivative with respect to time. Next task is to solve (2) and determine $\beta(t)$. In order to change (2) into the solvable differential equation , following the method of time-space transformation [16], if we perform the following time-space transformation:

$$y=z+\alpha(t),\qquad s=\int_0^t\frac{1}{m(\tau)}d\tau, \tag{3}$$

where $\alpha(t)$ is undetermined function of time, from (3) we can obtain

$$\frac{\partial}{\partial t}=\frac{1}{m}\frac{\partial}{\partial s}+\dot{\alpha}\frac{\partial}{\partial y},\qquad \frac{\partial}{\partial z}=\frac{\partial}{\partial y},\qquad \frac{\partial^2}{\partial z^2}=\frac{\partial^2}{\partial y^2}. \tag{4}$$

Making use of (4) and $\varphi(z,t)=f(y,s)$, then (2) can be easily written in form:

$$\begin{aligned}\frac{i}{m}\frac{\partial f(y,s)}{\partial s}&=-\frac{1}{2m}\frac{\partial^2 f(y,s)}{\partial y^2}+[g_1+\dot{\beta}]yf(y,s)-i\left\{\frac{\beta}{m}+g_2+\dot{\alpha}\right\}\frac{\partial f(y,s)}{\partial y}\\&\quad+\left\{\frac{\beta^2}{2m}+g_2\beta+g_3\right\}f(y,s)-\{g_1+\dot{\beta}\}\alpha f(y,s).\end{aligned} \tag{5}$$

In the present time-dependent potential model, due to $g_j(t)$ being undetermined function of time we can make $\beta(t)$ and $\alpha(t)$ to satisfy the following equations:

$$\dot{\alpha}+\frac{\beta}{m}+g_2=0,\qquad g_1+\dot{\beta}=0. \tag{6}$$

With the aid of (6), it is straightforward that (5) will become

$$\frac{i}{m}\frac{\partial f}{\partial s}=-\frac{1}{2m}\frac{\partial^2 f}{\partial y^2}+\left\{\frac{\beta^2}{2m}+g_2\beta+g_3\right\}f. \tag{7}$$

In order to delete the term $\{\frac{\beta 2}{2m} + g_2\beta + g_3\}f$ in (7), we assume:

$$f(y,s) = F(y,s)\exp\left[-i\int_0^t G(\tau)d\tau\right], \tag{8}$$

with $G = \frac{\beta^2}{2m} + \beta g_2 + g_3$. To make $f(y,s)$ being an analytical function, we must guarantee the integrals $\int_0^t G(\tau)d\tau$ and $s = \int_0^t \frac{1}{m(\tau)}d\tau$ to be integrable. In other words, $m(t)$ and $g_j(t)$ $(j = 1, 2, 3)$ can not be arbitrary function of time. Using (4) and (8), then the left side of (7) can be written as:

$$\begin{aligned}
\frac{i}{m}\frac{\partial f(y,s)}{\partial s} &= \frac{i}{m}\frac{\partial}{\partial s}\left\{F(y,s)\exp\left[-i\int_0^t G(\tau)d\tau\right]\right\}\\
&= \frac{i}{m}\frac{\partial F(y,s)}{\partial s}\exp\left[-i\int_0^t G(\tau)d\tau\right] + \frac{iF(y,s)}{m}\frac{\partial}{\partial s}\exp\left[-i\int_0^t G(\tau)d\tau\right]\\
&= \frac{i}{m}\frac{\partial F(y,s)}{\partial s}\exp\left[-i\int_0^t G(\tau)d\tau\right]\\
&\quad + iF(y,s)\left[\frac{\partial}{\partial t} - \dot{\alpha}\frac{\partial}{\partial y}\right]\exp\left[-i\int_0^t G(\tau)d\tau\right]\\
&= \left\{\frac{i}{m}\frac{\partial}{\partial s}F(y,s) + G(t)F(y,s)\right\}\exp\left[-i\int_0^t G(\tau)d\tau\right].
\end{aligned} \tag{9}$$

Substitute (9) and (8) to (7), then this term $\{\frac{\beta^2}{2m} + g_2\beta + g_3\}f$ in (7) will be deleted. Finally we get equation:

$$i\frac{\partial}{\partial s}F(y,s) = -\frac{1}{2}\frac{\partial^2}{\partial y^2}F(y,s), \tag{10}$$

which is the same as the one-dimensional Schrödinger equation of a free particle with mass $m = 1$ and $\hbar = 1$.

Following the ordinary procedures employed in the usual textbook of quantum mechanics, the solution to (10) can be written in the form:

$$F(y,s) = (1/\sqrt{2\pi})\exp[i(Ay - A^2 s/2)], \tag{11}$$

where A is an arbitrary real number. Finally, reversing the above procedure, the solution to the Schrödinger equation of the time-dependent potential considered here can been written as

$$\begin{aligned}
\psi(z,t) &= f(y,s)\exp[iz\beta(t)]\\
&= F(s,y)\exp\left[-i\int_0^t G(\tau)d\tau\right]\exp[iz\beta(t)]\\
&= \frac{1}{\sqrt{2\pi}}\exp\{iA[z+\alpha(t)]\}\exp\left\{-\frac{iA^2}{2}\int_0^t \frac{d\tau}{m(\tau)}\right\}\exp\left\{-i\int_0^t G(\tau)d\tau\right\}\\
&\quad \times \exp[i\beta(t)z],
\end{aligned} \tag{12}$$

where

$$\alpha(t) = -\int_0^t \{\beta(\tau)/m(\tau) + g_2(\tau)\}d\tau,$$

$$G(\tau) = \frac{\beta^2(\tau)}{2m(\tau)} + \beta(\tau)g_2(\tau) + g_3(\tau),$$

$$\beta(t) = -\int_0^t g_1(\tau)d\tau.$$

It is obvious that from (12), one can easily find the analytical solution of Schrödinger equation (1) for specified $m(t)$ and $g_j(t)$.

3 Discussion

In order to further check above our results, let us discuss the several special cases. First, if the time-dependent potential is taken in form $V(z,t) = g_1(t)z$ (namely, $g_1(t) \neq 0$, $g_2(t) = g_3(t) = 0$), (12) will reduce to the following form:

$$\psi(z,t) = \frac{1}{\sqrt{2\pi}} \exp\{iA[z + \alpha(t)]\} \exp\left\{-\frac{iA^2}{2}\int_0^t \frac{d\tau}{m(\tau)}\right\} \exp\left[-i\int_0^t G(\tau)d\tau\right] \exp[i\beta(t)z], \tag{13}$$

where

$$\beta(t) = -\int_0^t g_1(\tau)d\tau, \qquad G(\tau) = \frac{\beta^2(\tau)}{2m(\tau)}, \qquad \alpha(t) = -\int_0^t \frac{1}{m(\tau)}\beta(\tau)d\tau.$$

It is clear that the solution (13) agrees with the result of [16]. This indicates that the result of [16] is a special result obtained here.

Secondly, if the external field vanishes (namely, $g_1(t) = g_2(t) = g_3(t) = 0$) and the particle mass is independent on time (namely, $m(t) = m_0$, with m_0 being a constant), (12) will become

$$\psi(z,t) = \frac{1}{\sqrt{2\pi}} \exp(iAz) \exp\left(-\frac{iA^2}{2m_0}t\right) = \frac{1}{\sqrt{2\pi}} \exp i(pz - Et), \tag{14}$$

where $p = A$, and $E = \frac{A^2}{2m_0} = \frac{p^2}{2m_0}$. It is every obvious that the solution (14) is the usual free particle plane wave with energy E and momentum p along direction z [19]. This is just the result that we except to get because the quantum system of time-dependent potential considered here will change to that of the free particle when the external field vanishes.

Finally, if the time-dependent potential is taken in form $V(\hat{p},t) = g_2(t)\hat{p}$ (namely, $g_1(t) = g_3(t) = 0$) we have $\beta = G = 0$. In the case, from (12) we have

$$\psi(z,t) = \frac{1}{\sqrt{2\pi}} \exp\left\{iA\left[z - \int_0^t g_2(\tau)d\tau\right]\right\} \exp\left\{-\frac{iA^2}{2}\int_0^t \frac{d\tau}{m(\tau)}\right\}. \tag{15}$$

Note that when $V(\hat{p},t) = g_2(t)\hat{p}$, (1) will reduce to the Schrödinger equation with vector potential $\vec{A} = (0, 0, A_z)$ and scalar potential ϕ:

$$A_z = -cm(t)g_2(t)/q, \qquad \phi = -m(t)g_2^2(t)/2q, \tag{16}$$

where c stands for the velocity of light and q for the charge.

So the wave function (15) can describe the motion of charged particle in an electromagnetic field where the vector potential $\vec{A}$ and scalar potential ϕ is taken in form of (16).

4 Conclusions

In summary, we have exactly solved the Schrödinger equation with the time-dependent potential $V(z, \hat{p}, t) = g_1(t)z + g_2(t)\hat{p} + g_3(t)$, by the method of time-space transformations, in $1 + 1$ dimensions. The corresponding wave function has been found and the solutions of three special cases have been discussed. The results show that when $g_2(t) = g_3(t) = 0$, our general solution will reduce to that of [16] and when $g_1(t) = g_2(t) = g_3(t) = 0$ to the solutions of free particle Schrödinger equation [19]. Moreover, when $g_1(t) = g_3(t) = 0$, the corresponding wave function can describe the motion of charged particle in an electromagnetic field.

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